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CONTRIBUTIONS TO ECONOMIC ANALYSIS

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ESSAYS IN INDEX NUMBER THEORY VOLUME 1

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Essays in Index Number Theory, Volume I
W.E. Diewert and A.O. Nakamura (Editors)

Chapter 13

Fisher Ideal Output, Input and Productivity Indexes Revisited*

W.E. Diewert

1. Introduction

The basic purpose of this paper is to explain how productivity change can be measured for a firm under ideal circumstances.

If a firm produces only one output and utilizes only one input during each accounting period, then it is straightforward to define the productivity change for the firm between two periods. Let $y^t > 0$ denote the quantity of output produced during period t and let $x^t > 0$ denote the quantity of input utilized by the firm during period t for $t = 0, 1$. Then the productivity change going from period 0 to 1 may be defined by:

$$Pr(x^0, x^1, y^0, y^1) \equiv (y^1/y^0)/(x^1/x^0); \tag{1}$$

i.e., the productivity change is the firm's output ratio divided by its input ratio. Thus if output grows faster (slower) than input, Pr will be greater than one (less than one) and we say that the firm has experienced a productivity improvement (decline).

However, all firms utilize more than one input and virtually all firms produce more than one output. A basic research question is: how can definition (1) be generalized to the case of a multiple output, multiple input firm?

We shall take several different approaches to answering the above question. In our first approach, we replace the output ratio by an output quantity index, $Q(p^0, p^1, y^0, y^1)$, and replace the input ratio by an input quantity index, $Q^*(w^0, w^1, x^0, x^1)$, where $p^t \equiv (p_1^t, \ldots, p_M^t)$ and $y^t \equiv (y_1^t, \ldots, y_M^t)$ are the output price and quantity vectors pertaining to period t and $w^t \equiv (w_1^t, \ldots, w_N^t)$ and $x^t \equiv (x_1^t, \ldots, x_N^t)$ are the input price and quantity vectors pertaining to

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period t for $t = 0, 1$. Note that the index number functions Q and Q^* depend on the relevant quantity vectors for the two periods but they also use the corresponding price vectors as a means of weighting the quantities.

The problem with this index number approach is that we must specify concrete functional forms for the quantity indexes Q and Q^*. Four commonly used functional forms for Q are the Laspeyres quantity index Q_L, the Paasche quantity index Q_P, the Fisher ideal quantity index Q_F and the translog or Törnqvist quantity index Q_T defined by (2) – (5) below:[1]

$$Q_L(p^0,p^1,y^0,y^1) \equiv p^0 \cdot y^1/p^0 \cdot y^0; \tag{2}$$

$$Q_P(p^0,p^1,y^0,y^1) \equiv p^1 \cdot y^1/p^1 \cdot y^0; \tag{3}$$

$$Q_F(p^0,p^1,y^0,y^1) \equiv (p^0 \cdot y^1 p^1 \cdot y^1/p^0 \cdot y^0 p^1 \cdot y^0)^{1/2}; \tag{4}$$

$$Q_T(p^0,p^1,y^0,y^1) \equiv \prod_{i=1}^{M} (y_i^1/y_i^0)^{(s_i^0+s_i^1)/2} \tag{5}$$

where $s_i^t \equiv p_i^t y_i^t/p^t \cdot y^t$ is the period t revenue share of output i. Note that $Q_F = (Q_L Q_P)^{1/2}$; i.e., the Fisher ideal quantity index is the geometric mean of the Laspeyres and Paasche quantity indexes. Input quantity indexes Q_L^*, Q_P^*, Q_F^* and Q_T^* can be defined in a manner analogous to (2) – (5), except that input prices and quantities, w^t and x^t, replace the output prices and quantities, p^t and y^t.

In Sections 2 and 3 below, we use the *test* or *axiomatic approach* to index number theory to determine the functional forms for Q and Q^*. This approach dates back to C.M. Walsh [1901] [1921] and Irving Fisher [1911] [1921] [1922] but in more recent times, the main contributors have been Eichhorn [1976] [1978b] and his co-workers; see Eichhorn and Voeller [1976] and Funke and Voeller [1978] [1979].

In Section 2, we list some twenty tests or mathematical properties that have been suggested as desirable for an output quantity index $Q(p^0,p^1,y^0,y^1)$ and in Section 3, we show that the Fisher ideal quantity index Q_F defined by (4) above is the unique function which satisfies all of these tests. Thus, our first approach to the measurement of productivity change in the multiple input and output case leads to the Fisher index of productivity change Pr_F defined by:

$$Pr_F(p^0,p^1,y^0,y^1,w^0,w^1,x^0,x^1) \equiv Q_F(p^0,p^1,y^0,y^1)/Q_F^*(w^0,w^1,x^0,x^1). \tag{6}$$

In the remainder of the paper, we consider economic approaches to the construction of input, output and productivity indexes. In economic approaches, the assumption of optimizing behavior is always used; i.e., we assume that the firm competitively minimizes costs, maximizes revenues or maximizes profits. In the test or axiomatic approach, no assumption about optimizing behavior is required and this can be an advantage of the approach.

Our first economic approach to the measurement of productivity change is contained in Sections 4 and 5. It is well known that the technology of a firm can be represented in several alternative ways; e.g., by production functions, variable profit functions or by distance functions.[2] In Section 5, we represent the technology of the firm in period t by a specific functional form for its variable profit function. We then show that the shift in the technology can be computed exactly using a function of observable prices and quantities — in fact, the Fisher productivity index defined by (6) does the job. In Section 4, we show that the specific functional forms for the profit functions used in Section 5 are generalizations of a functional form for a variable profit function that is *flexible*;[3] i.e., the functional form can provide a second order approximation to an arbitrary twice continuously differentiable variable profit function. Thus, the Fisher index of productivity change defined by (6) turns out to be a *superlative*[4] index of productivity change.

Sections 6 and 7 of the paper present economic justifications for the use of the Fisher output and input indexes, Q_F and Q_F^* respectively, that are analogous to the economic justifications for the use of the translog output and input indexes, Q_T and Q_T^*, that were presented by Caves, Christensen and Diewert [1982b; 1395–1401].

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[1] The Laspeyres quantity index matches up with the Paasche [1874] price index while the Paasche quantity index matches up with the Laspeyres [1871] price index. Bowley [1921; 203] advocated the use of the Laspeyres quantity index to measure output change in the production context. The "ideal" properties of Q_F were stressed by Fisher [1922]. The quantity index Q_T is first mentioned in Fisher [1922; 473] where it is the quantity index which corresponds to price index number 124, but Törnqvist [1936] was the first to stress its favorable properties. Christensen and Jorgenson [1970] used Q_T and Q_T^* as discrete time approximations to the continuous time Divisia [1926] indexes while Solow [1957] used Q_L^* as a discrete approximation to the Divisia index of inputs. Diewert [1976a; 118–120] showed that Q_T corresponded to a translog aggregator function introduced by Christensen, Jorgenson and Lau [1971] and so Jorgenson and Nishimizu [1978] called Q_T the translog quantity index. Notation: $p \cdot y = p^T y \equiv \sum_{m=1}^{M} p_m y_m$.

[2] See Gorman [1968b], Diewert [1973a], Blackorby, Primont and Russell [1978] and McFadden [1978].

[3] The term "flexible" is due to Diewert [1974a; 113].

[4] Diewert [1976a; 117] used the term "superlative" to describe an index number formula that was exact for a flexible functional form.

In Section 8, a similar economic approach to productivity indexes is developed, except that the firm's technology is represented by an output distance or deflation function instead of a variable profit function. Again we obtain a strong economic justification for the use of the Fisher productivity index (6).

The above economic approaches to productivity indexes, relying on the theory of exact and superlative indexes, are analogous to the approaches used by Diewert [1976a; 124–127][5] [1980; 491–493] [1983b; 1077–1083] and Caves, Christensen and Diewert [1982b; 1401–1408]. In fact, under the hypothesis of a constant returns to scale technology, Caves, Christensen and Diewert [1982b; 1406] present a strong economic justification for the use of the following Törnqvist or translog productivity index to measure the shift in the technology:

$$(7)\quad Pr_T(p^0,p^1,y^0,y^1,w^0,w^1,x^0,x^1) \equiv Q_T(p^0,p^1,y^0,y^1)/Q_T^*(w^0,w^1,x^0,x^1)$$

where Q_T is the translog output index defined by (5) and Q_T^* is the analogous translog input index. The productivity change index (7) has been used by Christensen and Jorgenson [1970], Jorgenson and Griliches [1972] and the Bureau of Labor Statistics in their recent work on multifactor productivity; see Mark and Waldorf [1983; 15]. The results in Sections 5 and 8 below present equally strong economic justifications for the use of the Fisher productivity index Pr_F defined by (6).

Since Pr_T and Pr_F have equally valid economic justifications, the results of Section 3 nudge us in the direction of preferring Pr_F over Pr_T, since from the viewpoint of the test approach to index numbers, the Fisher index Q_F seems preferable to the translog index Q_T.

Section 9 concludes.

2. The Axiomatic or Test Approach to Index Number Theory

As we mentioned in the introduction, we want to use the test approach to index number theory in an attempt to determine the functional forms for the quantity indexes Q and Q^*. We shall concentrate our attention on the determination of the output index $Q(p^0,p^1,y^0,y^1)$; the theory for the input quantity index Q^* is analogous.

Instead of trying to directly determine the functional form for the quantity index Q, it turns out to be more convenient to *simultaneously* determine Q

[5]This approach was not satisfactory since it assumed separability between inputs and outputs. For discussions on separability concepts, see Blackorby, Primont and Russell [1978].

and its corresponding price index $P(p^0,p^1,y^0,y^1)$. Thus we want to find two functions of the $4M$ variables, $p^0 \equiv (p_1^0,\ldots,p_M^0)$, $p^1 \equiv (p_1^1,\ldots,p_M^1)$, $y^0 \equiv (y_1^0,\ldots,y_M^0)$ and $y^1 \equiv (y_1^1,\ldots,y_M^1)$ (which are the price and quantity vectors pertaining to periods 0 and 1), $P(p^0,p^1,y^0,y^1)$ and $Q(p^0,p^1,y^0,y^1)$, which decompose the value change between the two periods, $p^1 \cdot y^1/p^0 \cdot y^0$, into a price change part P and a quantity change part Q; i.e., we want P and Q to satisfy the following equation:[6]

$$(8)\qquad P(p^0,p^1,y^0,y^1)Q(p^0,p^1,y^0,y^1) = p^1 \cdot y^1/p^0 \cdot y^0.$$

If $M = 1$, so that there is only one output, then a natural candidate for P is p_1^1/p_1^0, the single price ratio, and a natural candidate for Q is y_1^1/y_1^0, the single quantity ratio.

Note that if either P or Q is determined, then the remaining function Q or P may be defined implicitly or residually using equation (8). Since historically researchers first concentrated on the determination of P, we shall also attempt to determine the functional form for P with the understanding that once P has been determined, Q may be determined using (8).

What index number theorists have done over the years is propose properties or tests that P should satisfy. These properties are generally multidimensional analogues to the one good price index formula, p_1^1/p_1^0. Below, we list twenty tests along with the names of the researchers who have proposed the corresponding tests.

We shall assume that every component of each price and quantity vector is positive; i.e., $p^t \gg 0_M$ and $y^t \gg 0_M$ for $t = 0,1$. If we want to set $y^0 = y^1$, we shall call the common quantity vector y; if we want to set $p^0 = p^1$, we call the common price vector p.

PT1: *Positivity*: $P(p^0,p^1,y^0,y^1) > 0$. Eichhorn and Voeller [1976; 23] suggested this test.

PT2: *Continuity*: $P(p^0,p^1,y^0,y^1)$ is a continuous function of its arguments.

This test does not seem to have been formally suggested in the literature.[7] However, Irving Fisher [1922; 207–215] seems to have informally suggested the essence of this test.[8]

[6]Frisch [1930; 399] called (8) the *product test*. The concept of this test was due to Fisher [1911; 418].

[7]Frisch [1930; 400] assumed differentiability which implies continuity. Also Eichhorn [1978b; 165] made a weaker continuity assumption.

[8]"A formula which can be shown to be especially erratic, as compared with other formulae, has been called freakish" (Fisher [1922; 207]). Also Fisher [1922; 114–115] disapproved of medians and modes because of their insensitivity to possibly large changes in the data and their violent change to possibly small changes in the data.

PT3: *Identity* or *Constant Prices Test*: $P(p, p, y^0, y^1) = 1$; i.e., if the price of every good is identical during the two periods, then the price index should equal unity, no matter what the quantity vectors are. Laspeyres [1871; 308], Walsh [1901; 308] and Eichhorn and Voeller [1976; 24] have all suggested this test.

PT4: *Tabular Standard, Basket* or *Constant Quantities Test*: $P(p^0, p^1, y, y) = p^1 \cdot y / p^0 \cdot y$; i.e., if quantities are constant during the two periods so that $y^0 = y^1 = y$, then the price index should equal the expenditure on the constant basket in period 1, $p^1 \cdot y$, divided by the expenditure on the basket in period 0, $p^0 \cdot y$.

The origins of this test go back at least two hundred years to the Massachusetts legislature which used a constant basket of goods to index the pay of Massachusetts soldiers fighting in the American Revolution; see Willard Fisher [1913]. Other researchers who have suggested the test over the years include: Lowe [1823; Appendix, 95], Scrope [1833; 406], Sidgwick [1883; 67–68], Jevons [1884; 122] originally published in 1865, Edgeworth [1925; 215] originally published in 1887, Marshall [1887; 363], Pierson [1895; 332] Walsh [1901; 540] [1921; 544], Bowley [1901; 227], Pigou [1912; 38], Frisch [1936; 6], Vogt [1978; 132] and Funke [1988; 103]. Scrope [1833; 407] was the first to use the term "tabular standard" while Edgeworth [1925; 331] used the term "consumption standard" to describe the test. Vogt [1978; 132] called the test the "Wertindextreue Test" and Funke [1988; 103] translated this German terminology into the "Value Ratio Preserving Test."

PT5: *Proportionality in Current Prices*: $P(p^0, \lambda p^1, y^0, y^1) = \lambda P(p^0, p^1, y^0, y^1)$ for scalars $\lambda > 0$; i.e., if all period 1 prices are multiplied by the positive number λ, then the new price index is λ times the old price index.

This test was proposed by Walsh [1901; 385], Eichhorn and Voeller [1976; 24] and Vogt [1980; 68].

Walsh [1901] and Irving Fisher [1911; 418] [1922; 420] proposed the related proportionality test $P(p, \lambda p, y^0, y^1) = \lambda$. This last test is a combination of PT3 and PT5; in fact Walsh [1901; 385] noted that this last test implies the identity test, PT3.

PT6: *Inverse Proportionality in Base Prices* (Homogeneity of Degree Minus One in Base Prices): $P(\lambda p^0, p^1, y^0, y^1) = \lambda^{-1} P(p^0, p^1, y^0, y^1)$ for all $\lambda > 0$; i.e., if all period 0 prices are multiplied by the positive number λ, then the new price index equals the old price index divided by λ. Eichhorn and Voeller [1976; 28] suggested this test.

The next seven tests are *invariance* or *symmetry* tests. Fisher [1922; 62–63 and 458–460] and Walsh [1921; 542] seem to have been the first researchers to appreciate the significance of these kinds of tests. Fisher [1922; 62–63] spoke of *fairness*, but it is clear that he had symmetry properties in mind. It is perhaps unfortunate that he did not realize that there were more symmetry

and invariance properties than the ones he proposed; if he had realized this, it is likely that he would have been able to characterize axiomatically his ideal price index, as we shall do below in Section 3.

PT7: *Invariance to Proportional Changes in Current Quantities* (Homogeneity of Degree Zero in Current Quantities): $P(p^0, p^1, y^0, \lambda y^1) = P(p^0, p^1, y^0, y^1)$ for all $\lambda > 0$; i.e., if current period quantities are all multiplied by the number λ, then the price index remains unchanged. Vogt [1980; 70] seems to have been the first to propose this test and his derivation of the test is of some interest. Recall the product test, equation (8) above. Suppose the quantity index Q satisfies the quantity analogue to the price test PT5; i.e., suppose Q satisfies $Q(p^0, p^1, y^0, \lambda y^1) = \lambda Q(p^0, p^1, y^0, y^1)$. Then using (8), we see that $P(p^0, p^1, y^0, y^1)$ must satisfy PT7.

Vogt [1980; 70] used the same type of argument to derive PT4 as a consequence of the quantity index satisfying an analogue to the identity test PT3; i.e., suppose Q satisfies $Q(p^0, p^1, y, y) = 1$. Then by (8), $P(p^0, p^1, y, y) = p^1 \cdot y / p^0 \cdot y Q(p^0, p^1, y, y) = p^1 \cdot y / p^0 \cdot y$ which is PT4.[9] Thus if a quantity index Q satisfies a certain test or property, then (8) may be used to deduce the corresponding property or test that the price index P must satisfy. This observation was first made by Irving Fisher [1911; 400–406] in an almost forgotten (but nonetheless brilliant) work.

PT8: *Invariance to Proportional Changes in Base Quantities* (Homogeneity of Degree Zero in Base Quantities): $P(p^0, p^1, \lambda y^0, y^1) = P(p^0, p^1, y^0, y^1)$ for all $\lambda > 0$; i.e., if all base period quantities are multiplied by the number λ, then the price index remains unchanged.

Surprisingly, this test does not seem to have been proposed before.

If the quantity index Q satisfies the following counterpart to PT6: $Q(p^0, p^1, \lambda y^0, y^1) = \lambda^{-1} Q(p^0, p^1, y^0, y^1)$ for all $\lambda > 0$, then using (8), the corresponding price index P must satisfy: $P(p^0, p^1, \lambda y^0, y^1) = p^1 \cdot y^1 / p^0 \cdot \lambda y^0 Q(p^0, p^1, \lambda y^0, y^1) = p^1 \cdot y^1 / p^0 \cdot \lambda y^0 \lambda^{-1} Q(p^0, p^1, y^0, y^1) = P(p^0, p^1, y^0, y^1)$; i.e., P must satisfy

[9] This derivation may be found in Irving Fisher [1911; 401] if we set his $k^1 = 1$. The complete derivation of Fisher's Test 2 proceeded as follows. Suppose the quantity index satisfies the following Proportionality Test: $Q(p^0, p^1, y, k^1 y) = k^1$. Then P must satisfy: $P(p^0, p^1, y, k^1 y) = p^1 \cdot k^1 y / p^0 \cdot y Q(p^0, p^1, y, k^1 y) = p^1 \cdot k^1 y / p^0 \cdot y k^1 = p^1 \cdot y / p^0 \cdot y$. Thus if Q satisfies Fisher's Proportionality Test, then the corresponding P must satisfy $P(p^0, p^1, y, \lambda y) = p^1 \cdot y / p^0 \cdot y$ for all $\lambda > 0$, which is a stronger version of the Basket Test, PT4. Fisher [1911; 406] thought that this price test was the most important of his eight tests for prices because it was the only test that indicated what type of quantity weighting of the prices was required. However, later Fisher [1922; 420–421] no longer seemed to consider that the test was important. Note that we can now interpret PT4 as an implicit identity test.

PT8. This argument provides some justification for assuming the validity of PT8.

PT7 and PT8 together impose the property that the price index P does not depend on the absolute magnitudes of the quantity vectors y^0 and y^1. Of course, in the one good case, PT4 and either PT7 or PT8 implies that $P(p_1^0, p_1^1, y_1^0, y_1^1) = p_1^1/p_1^0$, the price ratio for the single good.

PT9: *Commodity Reversal Test* (Invariance to Changes in the Ordering of Commodities): $P(\tilde{p}^0, \tilde{p}^1, \tilde{y}^0, \tilde{y}^1) = P(p^0, p^1, y^0, y^1)$ where $\tilde{p}^t$ denotes a permutation of the components of the vector p^t and $\tilde{y}^t$ denotes the same permutation of the components of y^t for $t = 0, 1$. Thus if the ordering of the commodities is changed, the numerical value of the price index remains unchanged.

This test is due to Irving Fisher [1922][10] and it is one of his three famous reversal tests. The other two are the time reversal test and the factor reversal test which will be considered below.

PT10: *Invariance to Changes in the Units of Measurement* (Commensurability Test): $P(\alpha_1 p_1^0, \ldots, \alpha_M p_M^0;\ \alpha_1 p_1^1, \ldots, \alpha_M p_M^1;\ \alpha_1^{-1} y_1^0, \ldots, \alpha_M^{-1} y_M^0;\ \alpha_1^{-1} y_1^1, \ldots, \alpha_M^{-1} y_M^1) = P(p_1^0, \ldots, p_M^0;\ p_1^1, \ldots, p_M^1;\ y_1^0, \ldots, y_M^0;\ y_1^1, \ldots, y_M^1)$ for all $\alpha_1 > 0, \ldots, \alpha_M > 0$; i.e., the price index does not change if the units of measurement for each commodity are changed.

The concept of this test was due to Jevons [1884; 23] and the Dutch economist Pierson [1896; 131], who criticized several index number formula for not satisfying this fundamental test. Fisher [1911; 411] first called this test the change of units test and later Fisher [1922; 420] called it the commensurability test.

PT11: *Time Reversal Test*: $P(p^1, p^0, y^1, y^0) = 1/P(p^0, p^1, y^0, y^1)$; i.e., if the data for periods 0 and 1 are interchanged, then the resulting price index should equal the reciprocal of the original price index. Obviously, in the one good case when the price index is simply the single price ratio, this test is satisfied (as are all of the other tests listed in this section).

When the number of goods is greater than one, many commonly used price indexes fail this test; e.g., the Laspeyres [1871] price index, $P_L \equiv p^1 \cdot y^0 / p^0 \cdot y^0$, and the Paasche [1874] price index, $P_P \equiv p^1 \cdot y^1 / p^0 \cdot y^1$, both fail this fundamental test.

The concept of the test was due to Pierson [1896; 128], who was so upset with the fact that many of the commonly used index number formulae did not satisfy this test that he proposed that the entire concept of an index number should be abandoned. More formal statements of the test were made by Walsh [1901; 368] [1921; 541] and Fisher [1911; 534] [1922; 64].

[10] Fisher [1922; 63] comments on this test that: "This is so simple as never to have been formulated."

Our next two tests are more controversial, since they do not appear to be consistent with the economic approach to index number theory.[11]

PT12: *Quantity Reversal Test* (Quantity Weights Symmetry Test): $P(p^0, p^1, y^0, y^1) = P(p^0, p^1, y^1, y^0)$; i.e., if the quantity vectors for the two periods are interchanged, then the price index remains invariant. This property means that if quantities y_m^t are used to weight the prices in the index number formula, then y_m^0 and y_m^1 must enter the formula in a symmetric manner; i.e., quantities from each period enter the formula symmetrically.

Funke and Voeller [1978; 3] introduced this test; they called it the "weight property".

PT13: *Price Reversal Test* (Price Weights Symmetry Test): $p^1 \cdot y^1 / p^0 \cdot y^0 P(p^0, p^1, y^0, y^1) = p^0 \cdot y^1 / p^1 \cdot y^0 P(p^1, p^0, y^0, y^1)$. This test has not been proposed before[12] but it is the analogue to PT12 applied to quantity indexes; i.e., if we use (8) to define the quantity index Q in terms of the price index P, then it can be seen that PT13 is equivalent to the following property for the associated quantity index Q:

$$Q(p^0, p^1, y^0, y^1) = Q(p^1, p^0, y^0, y^1); \tag{9}$$

i.e., if the price vectors for the two periods are interchanged, then the quantity index remains invariant. Thus if prices for the same good in the two periods are used to weight quantities in the construction of the quantity index, then property PT13 implies that these prices enter the quantity index in a symmetric manner.

The next three tests are mean value tests.

PT14: *Mean Value Test for Prices*:

$$\begin{aligned}\min_i \{p_i^1/p_i^0 : i = 1, \ldots, M\} &\le P(p^0, p^1, y^0, y^1) \\ &\le \max_i \{p_i^1/p_i^0 : i = 1, \ldots, M\};\end{aligned}$$

i.e., the price index lies between the minimum price ratio and the maximum price ratio. This very desirable property seems to have been first proposed by Eichhorn and Voeller [1976; 10].

PT15: *Mean Value Test for Quantities*:

$$\begin{aligned}\min_i \{y_i^1/y_i^0 : i = 1, \ldots, M\} &\le p^1 \cdot y^1 / p^0 \cdot y^0 P(p^0, p^1, y^0, y^1) \\ &\le \max_i \{y_i^1/y_i^0 : i = 1, \ldots, M\}.\end{aligned}$$

[11] See the critical discussion by Sato [1980].

[12] Funke and Voeller [1979; 55] proposed the following (different) price reversal test: $P(p^0, p^1, y^0, y^1) P(p^1, p^0, y^0, y^1) = 1$. Sato [1980; 127] showed that this price reversal test along with the time reversal test PT11 implied the quantity reversal test PT12.

Using (8) to define the quantity index Q in terms of the price index P, we see that PT15 is equivalent to the following property on the associated quantity index:

$$\text{(10)}\qquad \begin{aligned}\min_i\{y_i^1/y_i^0 : i=1,\ldots,M\} &\leq Q(p^0,p^1,y^0,y^1)\\ &\leq \max_i\{y_i^1/y_i^0 : i=1,\ldots,M\};\end{aligned}$$

i.e., the implicit quantity index Q defined by P lies between the minimum and maximum rates of growth of the individual quantities.

This test does not seem to have been proposed before, but it is an obvious quantity analogue to the price test PT14.

PT16: *Paasche and Laspeyres Bounding Test*: The price index P satisfies at least one of the following inequalities (11) or (12):

$$\text{(11)}\qquad p^1\cdot y^0/p^0\cdot y^0 \leq P(p^0,p^1,y^0,y^1) \leq p^1\cdot y^1/p^0\cdot y^1;$$

$$\text{(12)}\qquad p^1\cdot y^1/p^0\cdot y^1 \leq P(p^0,p^1,y^0,y^1) \leq p^1\cdot y^0/p^0\cdot y^0;$$

i.e., the price index P must lie between the Laspeyres and Paasche price indexes.

A justification for this test can be made by considering the basket test PT4. If $y^0 = y^1$, then by PT4, the correct functional form for the price index is $p^1\cdot y^0/p^0\cdot y^0 = p^1\cdot y^1/p^0\cdot y^1$. In the general case where $y^0 \neq y^1$, it is natural to think of both y^0 and y^1 as being "extreme" baskets and so the Laspeyres and Paasche price indexes should provide bounds to the "best" price index $P(p^0,p^1,y^0,y^1)$ which treats the quantity data in each period in a symmetric manner instead of the extreme manner implied by the Paasche and Laspeyres price indexes, which each use the quantity data for only one period.

In the context of the axiomatic or test approach to index number theory, PT16 has been proposed by both Bowley [1901; 227] and Fisher [1922; 403].[13]

We could propose a test where the implicit quantity index Q that corresponds to P via (8) is to lie between the Laspeyres and Paasche quantity indexes, Q_L and Q_P, defined by (2) and (3) above. However, the resulting test turns out to be equivalent to (11) and (12).[14]

Our final four tests are monotonicity tests.

PT17: *Monotonicity in Current Prices*: $P(p^0,p^1,y^0,y^1) < P(p^0,p,y^0,y^1)$ if $p^1 < p$; i.e., if some period 1 price increases, then the price index must increase, so that $P(p^0,p^1,y^0,y^1)$ is increasing in the components of p^1. This property was proposed by Eichhorn and Voeller [1976; 23].

PT18: *Monotonicity in Base Prices*: $P(p^0,p^1,y^0,y^1) > P(p,p^1,y^0,y^1)$ if $p^0 < p$; i.e., if any period 0 price increases, then the price index must decrease, so that $P(p^0,p^1,y^0,y^1)$ is decreasing in the components of p^0. This property was also proposed by Eichhorn and Voeller [1976; 23].

PT19: *Monotonicity in Current Quantities*: $p^1\cdot y^1/p^0\cdot y^0 P(p^0,p^1,y^0,y^1) < p^1\cdot y/p^0\cdot y^0 P(p^0,p^1,y^0,y)$ if $y^1 < y$.

PT20: *Monotonicity in Base Quantities*: $p^1\cdot y^1/p^0\cdot y^0 P(p^0,p^1,y^0,y^1) > p^1\cdot y^1/p^0\cdot y P(p^0,p^1,y,y^1)$ if $y^0 < y$.

If we define the implicit quantity index Q that corresponds to P using (8), we find that PT19 translates into the following inequality involving Q:

$$\text{(13)}\qquad Q(p^0,p^1,y^0,y^1) < Q(p^0,p^1,y^0,y) \quad \text{if} \quad y^1 < y;$$

i.e., if any period 1 quantity increases, then the quantity index must increase.

Similarly, we find that PT20 translates into:

$$\text{(14)}\qquad Q(p^0,p^1,y^0,y^1) > Q(p^0,p^1,y,y^1) \quad \text{if} \quad y^0 < y;$$

i.e., if any period 0 quantity increases, then the quantity index must decrease.

Tests PT19 and PT20 are due to Vogt [1980; 70].

This concludes our listing of tests. In the next section, we ask whether any index number formula $P(p^0,p^1,y^0,y^1)$ exists that can satisfy all twenty tests.

3. The Test Approach and Fisher Ideal Index Numbers

Recall the twenty tests listed in the previous section. Our main result is Theorem 1.

THEOREM 1. *The only index number formula $P(p^0,p^1,y^0,y^1)$ which satisfies tests PT1 – PT20 is the Fisher ideal price index P_F defined below by (16).*

Proof: Using the price reversal test, PT13, as well as the positivity test, PT1, we may rearrange terms to obtain the following equality:

$$\text{(15)}\qquad \begin{aligned} p^1\cdot y^1 p^1\cdot y^0/p^0\cdot y^0 p^0\cdot y^1 &= P(p^0,p^1,y^0,y^1)/P(p^1,p^0,y^0,y^1)\\ &= P(p^0,p^1,y^0,y^1)/P(p^1,p^0,y^1,y^0)\\ &\qquad\text{using PT12, the quantity reversal test,}\\ &= P(p^0,p^1,y^0,y^1)P(p^0,p^1,y^0,y^1)\\ &\qquad\text{using PT11, the time reversal test.}\end{aligned}$$

[13] Of course, the Paasche and Laspeyres price indexes arise repeatedly as bounds to the true index in the economic theory of index numbers; e.g., see Fisher and Shell [1972b; 57–58], Hicks [1940] or Diewert [1983b].

[14] Thus PT16 can be given a quantity index justification.

The desired result now follows by taking the positive square root of both sides of (15); i.e., we obtain:

(16) $P(p^0,p^1,y^0,y^1) = [(p^1\cdot y^0/p^0\cdot y^0)(p^1\cdot y^1/p^0\cdot y^1)]^{1/2} = (P_L P_P)^{1/2} \equiv P_F$

where $P_L \equiv p^1\cdot y^0/p^0\cdot y^0$ is the Laspeyres price index and $P_P \equiv p^1\cdot y^1/p^0\cdot y^1$ is the Paasche price index.

Note that we established (16) using only PT1 and the three reversal tests PT11, PT12 and PT13.

It can be verified by direct computations that P_F satisfies the remaining 16 tests. Q.E.D.

Of course, the quantity index that corresponds to the Fisher price index using the product test (8) is Q_F, the Fisher quantity index, defined by (4).

It turns out that P_F satisfies yet another test, PT21, which was Irving Fisher's [1921; 534] [1922; 72–81] third reversal test (the other two being PT9 and PT11):

PT21: *Factor Reversal Test* (Functional Form Symmetry Test): $P(p^0,p^1,y^0,y^1)P(y^0,y^1,p^0,p^1) = p^1\cdot y^1/p^0\cdot y^0$. A justification for this test is the following one: if $P(p^0,p^1,y^0,y^1)$ is a good functional form for the price index, then if we reverse the roles of prices and quantities, $P(y^0,y^1,p^0,p^1)$ ought to be a good functional form for a quantity index (which seems to be a correct argument) and thus the product of the price index $P(p^0,p^1,y^0,y^1)$ and the quantity index $P(y^0,y^1,p^0,p^1)$ ought to equal the value ratio, $p^1\cdot y^1/p^0\cdot y^0$. The second part of this argument does not seem to be valid and thus many researchers over the years have objected to the factor reversal test.[15] However, if one is willing to embrace PT21 as a basic test, Funke and Voeller [1978; 180] obtained the following result:

THEOREM 2. *(Funke and Voeller): The only index number function $P(p^0,p^1,y^0,y^1)$ which satisfies PT1 (positivity), PT11 (time reversal test), PT12 (quantity reversal test) and PT21 (factor reversal test) is the Fisher ideal index P_F defined by (16).*

Our proof of Theorem 1 is very similar to Funke and Voeller's proof of Theorem 2.[16]

The Fisher price index P_F satisfies all 20 of the tests listed in the previous section. Which tests do other commonly used price indexes satisfy? Recall Q_L, Q_P and Q_T defined by (2), (3) and (5) above. The corresponding price indexes defined using (8) are:

(17) $$p^1\cdot y^1/p^0\cdot y^0 Q_L = p^1\cdot y^1/p^0\cdot y^1 \equiv P_P(p^0,p^1,y^0,y^1),$$

(18) $$p^1\cdot y^1/p^0\cdot y^0 Q_P = p^1\cdot y^0/p^0\cdot y^0 \equiv P_L(p^0,p^1,y^0,y^1), \text{ and}$$

(19) $$p^1\cdot y^1/p^0\cdot y^0 Q_T = p^1\cdot y^1/p^0\cdot y^0 \prod_{i=1}^M (y_i^1/y_i^0)^{(s_i^0+s_i^1)/2} \equiv \tilde{P}_T,$$

where P_P, P_L and $\tilde{P}_T$ are the Paasche, Laspeyres and implicit translog price indexes respectively. Two additional price indexes that are often used in productivity studies are P_T, the direct translog price index, and P_W, the Walsh [1901; 373] price index,[17] defined as follows:

(20) $$P_T(p^0,p^1,y^0,y^1) \equiv \prod_{i=1}^M (p_i^1/p_i^0)^{(s_i^0+s_i^1)/2},$$

(21) $$P_W(p^0,p^1,y^0,y^1) \equiv \sum_{i=1}^M p_i^1(y_i^1 y_i^0)^{1/2} \Big/ \sum_{j=1}^M p_j^0(y_j^1 y_j^0)^{1/2},$$

where $s_i^t \equiv p_i^t y_i^t/p^t\cdot y^t$ is the expenditure share of good i in period t as usual.

Straightforward computations show that the Paasche and Laspeyres price indexes, P_P and P_L, fail only the three reversal tests, PT11, PT12 and PT13.

[15] Bowley [1923; 93] objected to PT21 on statistical grounds: "For (2), there seems to be no justification on general principles; the mean of a product only equals the product of the means of its factors if there is no correlation between them." The next objector to PT21 was Davies [1924; 187]: "That price and quantity each requires a distinct type of formula is indicated by the simpler problem where only one commodity is involved, as in the case of the bushels of wheat previously discussed. In this case, the measure of quantities for each period is obviously obtained merely by summing up the number of units sold, while the measure of prices is obtained by dividing the aggregate value by the quantity units. It would therefore be expected that the composite indexes derived by an analogous method would be associated with distinctive formulae for prices and quantities. Hence the factor reversal test may be disregarded." Note that Davies attacked PT21 from the perspective of what Frisch [1930; 397] called the theory of ***absolute index numbers*** which looks for functions P and Q which satisfy the following modified product tests: $P(p^0)Q(y^0) = p^0\cdot y^0$ or $P(p^0,y^0)Q(p^0,y^0) = p^0\cdot y^0$. Eichhorn [1978b; 141–146] calls these absolute index numbers, P and Q, ***price levels*** and ***quantity levels***, and he finds, under certain regularity conditions, that functions P and Q satisfying the modified product tests do not exist. Frisch [1930; 398] calls functions P and Q that satisfy the usual product test (8) ***relative index numbers***. Finally, Samuelson and Swamy [1974; 575] criticize PT21 from the perspective of the economic theory of index numbers and conclude: "A man and wife should be properly matched; but that does not mean I should marry my identical twin!"

[16] For another axiomatic characterization of the Fisher ideal price index (and a history of the subject), see Balk [1985].

[17] Walsh [1901; 373] called his price index "Scrope's emended method."

Since the quantity and price reversal tests, PT12 and PT13, are somewhat controversial, the test performance of P_L and P_P seems quite good. However, the failure of the time reversal test, PT11, seems to be a fatal flaw associated with the use of these indexes.

The Walsh price index, P_W, fails only four tests: PT13, the price reversal test; PT16, the Paasche and Laspeyres bounding test; PT19, the monotonicity in current quantities test; and PT20, the monotonicity in base quantities test.

Finally, the translog price index P_T and the implicit translog price index $\tilde{P}_T$ each fail nine tests. Both indexes fail PT12, PT13, PT16 and the monotonicity tests PT17 to PT20. In addition, P_T fails PT4 (the basket test) and PT15 (the mean value test for quantities), while $\tilde{P}_T$ fails PT3 (the identity test) and PT14 (the mean value test for prices). Thus the translog indexes are subject to a rather high failure rate.

As we mentioned at the beginning of Section 2, our 20 tests on the price index function $P(p^0, p^1, y^0, y^1)$ are really 20 tests on P and the corresponding implicit quantity index $Q(p^0, p^1, y^0, y^1)$ which can be defined in terms of P using the product test (8).

The conclusion we draw from the results of this section is that from the viewpoint of the test approach to index numbers, the Fisher quantity index Q_F appears to be far superior to the translog quantity index Q_T. Hence from the viewpoint of the test approach, the Fisher productivity index Pr_F defined by (6) appears to be superior to the translog productivity index Pr_T defined by (7).

In the following sections of this paper, we will also provide economic justifications for the use of the Fisher productivity index Pr_F. In the following section, we start our discussion of the economic approach to index number theory by proving a flexibility theorem for a certain functional form. This functional form will play an important role in subsequent sections. However, the reader who is interested in productivity indexes can skip to Section 5.

4. A New Flexible Functional Form for a Revenue Function

Let $y \equiv (y_1, \ldots, y_M)$ be a nonnegative output vector and let $x \equiv (x_1, \ldots, x_N)$ be a nonnegative input vector. Then the technology of a firm that produces these M outputs and uses these N inputs can be represented by a production or transformation function f; i.e., $y_1 = f(y_2, \ldots, y_M, x)$ represents the maximum[18] amount of output 1 that can be produced using the vector of inputs x given that amounts $y_2, \ldots, y_M$ of outputs $2, \ldots, M$ must be produced.

[18]This maximum is conditional on current managerial knowledge and practices. If the output targets $y_2, \ldots, y_M$ are too high relative to the available amount of inputs $x \equiv (x_1, \ldots, x_N)$, then $f(y_2, \ldots, y_M, x) \equiv -\infty$.

Given a positive vector of output prices $p \equiv (p_1, \ldots, p_M) \gg 0_M$, define the firm's *revenue function*[19] as follows:

$$r(p, x) \equiv \max_y \{p \cdot y : y_1 = f(y_2, \ldots, y_M, x)\}; \tag{22}$$

i.e., the firm maximizes revenue $p \cdot y \equiv \sum_{m=1}^M p_m y_m$ subject to its production function constraint and the resulting maximized revenue is set equal to $r(p, x)$.

It is obvious that f completely determines r. Diewert [1973a] shows that under certain regularity conditions on f, r also completely determines f. It can also be shown that $r(p, x)$ will always be linearly homogeneous in the components of p for fixed x. If the production function f is linearly homogeneous, so that the technology is subject to constant returns to scale, then it can be shown[20] that $r(p, x)$ is linearly homogenous in the components of x for fixed p.

In what follows, we shall assume a constant returns to scale technology. However, this is not restrictive: if we want to model the nonconstant returns to scale case, we need only add an artificial fixed factor whose quantity is always set equal to 1; i.e., add $x_{N+1} \equiv 1$ to the other N inputs.[21]

In order for a functional form for a revenue function, $r(p, x)$, to be *flexible* at the point p^*, x^*, the functional form must have enough free parameters so that it can approximate an arbitrary twice continuously differentiable revenue function r^* to the second order of p^*, x^*; i.e., we require that r have enough free parameters so that the following equations can be satisfied:

$$r(p^*, x^*) = r^*(p^*, x^*); \tag{23}$$

$$\nabla_p r(p^*, x^*) = \nabla_p r^*(p^*, x^*); \tag{24}$$

$$\nabla_x r(p^*, x^*) = \nabla_x r^*(p^*, x^*); \tag{25}$$

$$\nabla^2_{pp} r(p^*, x^*) = \nabla^2_{pp} r^*(p^*, x^*); \tag{26}$$

$$\nabla^2_{xx} r(p^*, x^*) = \nabla^2_{xx} r^*(p^*, x^*); \tag{27}$$

$$\nabla^2_{px} r(p^*, x^*) = \nabla^2_{px} r^*(p^*, x^*); \tag{28}$$

where $\nabla_p r \equiv (\partial r/\partial p_1, \ldots, \partial r/\partial p_M)$ and $\nabla_x r \equiv (\partial r/\partial x_1, \ldots, \partial r/\partial x_N)$ are vectors of first order partial derivatives of r with respect to the components of

[19]The concept is due to Samuelson [1953–54; 20]. Gorman [1968b] uses the term "gross profit function" while McFadden [1978a] uses the term "conditional profit function" and Diewert [1973a] [1974a] uses the term "variable profit function."

[20]See Diewert [1973a; 291–294].

[21]The "price" of this input will be w_{N+1}, the pure profits (or losses) of the firm.

p and x respectively and $\nabla^2_{pp}r$, $\nabla^2_{xx}r$ and $\nabla^2_{px}r$ are matrices of second order partial derivatives of r with respect to the components of p and x. Taking into account the symmetry of the matrices $\nabla^2_{pp}r$ and $\nabla^2_{xx}r$, it appears that r would require at least $1 + M + N + (1/2)(M+1)M + (1/2)(N+1)N + MN$ parameters. However, this computation neglects the assumption that r and r^* are assumed to be linearly homogeneous in p and x separately.

If $r(p,x)$ is linearly homogeneous in p, then using Euler's theorem on homogeneous functions it can be shown that[22] r must satisfy the following $1 + M + N$ restrictions:[23]

(29) $$r(p^*, x^*) = p^{*T}\nabla_p r(p^*, x^*);$$

(30) $$\nabla^2_{pp} r(p^*, x^*)p^* = 0_M;$$

(31) $$p^{*T}\nabla^2_{px} r(p^*, x^*) = \nabla^T_x r(p^*, x^*).$$

Similarly, if $r(p,x)$ is linearly homogeneous in x, then it can be shown that r must satisfy the following $1 + N + M$ restrictions:

(32) $$r(p^*, x^*) = x^{*T}\nabla_x r(p^*, x^*);$$

(33) $$\nabla^2_{xx} r(p^*, x^*) = 0_N;$$

(34) $$\nabla^2_{px} r(p^*, x^*)x^* = \nabla_p r(p^*, x^*).$$

However, not all of the restrictions (29) to (34) are independent: if we post-multiply both sides of (31) by x^*, premultiply both sides of (34) by p^{*T} and use (29) and (32), we find that the last equation in (34) is implied by the other equations. Hence there are $2M + 2N + 1$ independent restrictions on the first and second partial derivations of r in (29)–(34). Note that r^* must also satisfy the restrictions (29)–(34).

Taking into account the restrictions (29)–(34), we see that in order for r to be flexible, it must have at least $1+M+N+(1/2)(M+1)M+(1/2)(N+1)N+MN-(2M+2N+1) = (1/2)M(M+1)+(1/2)N(N+1)+(M-1)(N-1)-1$ independent parameters.

Consider the following functional form for r:

(35) $$r(p,x) \equiv (p^T Apx^T Cx + \alpha^T p\beta^T xp^T Bx)^{1/2}, \quad A = A^T,\ C = C^T$$

where A, B and C are parameter matrices and α and β are parameter vectors. The following theorem shows that r is a flexible functional form.

[22] See Diewert [1973a; 308] or Diewert [1974a; 143–145].

[23] Notation: p^* and $\nabla_p\Pi$ are defined as column vectors and $p^{*T}\nabla_p\Pi$ denotes their inner product so p^{*T} is the transpose of p^*; 0_M denotes a vector of zeros of dimension M.

Theorem 3. *Let r^* be the revenue function that corresponds to a constant returns to scale technology. Suppose r^* is twice continuously differentiable at $p^* \gg 0_M$, $x^* \gg 0_N$ with $r^*(p^*, x^*) > 0$. Then for every α and β such that*

(36) $$\alpha^T p^* \neq 0, \quad \beta^T x^* \neq 0,$$

there exist symmetric matrices A and C and a matrix B such that

(37) $$p^{*T}B = 0^T_N, \quad Bx^* = 0_M$$

and r defined by (35) approximates r^* to the second order at (p^*, x^*). Moreover, the matrix A can be chosen to satisfy the following normalization:

(38) $$p^{*T}Ap^* = r^*(p^*, x^*) > 0.$$

Proof: We need to choose A, B and C so that equations (23)–(28) are satisfied. Note that $r(p,x)$ defined by (35) is linearly homogeneous in p for fixed x and linearly homogeneous in x for fixed p, as is $r^*(p,x)$. Thus both r and r^* satisfy the restrictions (29)–(34).

Define the vectors y^* and w^* as follows:

(39) $$y^* \equiv \nabla_p r^*(p^*, x^*); \qquad w^* \equiv \nabla_x r^*(p^*, x^*).$$

Using (29) and (32) applied to r^*, we have

(40) $$r^*(p^*, x^*) = p^{*T}y^* = w^{*T}x^*.$$

Now define A, B and C as follows:

(41) $$A \equiv \nabla^2_{pp} r^*(p^*, x^*) + (p^{*T}y^*)^{-1}y^*y^{*T};$$

(42) $$C \equiv \nabla^2_{xx} r^*(p^*, x^*) + (w^{*T}x^*)^{-1}x^*x^{*T};$$

(43) $$B \equiv 2(\alpha^T p^*\beta^T x^*)^{-1}[p^{*T}y^*\nabla^2_{px} r^*(p^*, x^*) - y^*w^{*T}].$$

Using properties (30) and (33) applied to r^*, we have:

(44) $$Ap^* = y^* \qquad \text{and}$$

(45) $$Cx^* = w^*.$$

Equations (40) and (44) may be used to show that A satisfies (38).

Now use (31) and (34) applied to r^*, (36), (39) and (40) to show that B defined by (43) satisfies the restrictions (37).

Premultiply both sides of (45) by x^{*T} and use (40) to show:

$$x^{*T}Cx^* = r^*(p^*, x^*). \tag{46}$$

Now use (35), (37), (38) and (46) to show that (23) is satisfied.
Differentiate r with respect to p and obtain:

$$\begin{aligned} \nabla_p r(p^*, x^*) &= [r(p^*, x^*)]^{-1} Ap^* x^{*T} Cx^* && \text{using (37)} \\ &= [r^*(p^*, x^*)]^{-1} Ap^* x^{*T} Cx^* && \text{using (23)} \\ &= Ap^* && \text{using (46)} \\ &= y^* && \text{using (44)} \\ &= \nabla_p r^*(p^*, x^*) && \text{using (39).} \end{aligned}$$

Thus equations (24) are satisfied. Similarly, differentiate r with respect to x and obtain

$$\begin{aligned} \nabla_x r(p^*, x^*) &= [r(p^*, x^*)]^{-1} p^{*T} Ap^* Cx^* && \text{using (37)} \\ &= Cx^* && \text{using (23) and (38)} \\ &= w^* && \text{using (45)} \\ &= \nabla_x r^*(p^*, x^*) && \text{using (39)} \end{aligned}$$

and thus equations (25) are also satisfied.

Finally, it can be shown that equations (41)–(43) imply equations (26)–(28). Q.E.D.

Thus for essentially arbitrary α and β vectors, the r defined by (35) is a flexible functional form. Note that given α and β, we have used a minimal number of parameters in the A, B and C matrices to achieve the flexibility result; i.e., taking into account the restriction (38), there are $(1/2)M(M+1)-1$ independent a_{ij} parameters, $(1/2)N(N+1)$ independent c_{ij} parameters and, taking into account the restrictions (37), $(M-1)(N-1)$ independent b_{ij} parameters. We shall use generalizations of this functional form in the next section.

5. An Economic Approach to Productivity Indexes

Assume that we can observe a firm's vector of outputs y^t, inputs x^t, output prices p^t and input prices w^t for periods $t = 0$ and $t = 1$. The firm's production function in period t is f^t and the corresponding dual revenue function is r^t defined by (22) (with r^t and f^t replacing r and f) for $t = 0, 1$.

In the economic approach to productivity measurement, a change in productivity is taken to be a shift in the production function or in one of the dual representations of the production function such as the revenue function.[24]

In this section, we shall identify a productivity change with a shift in the firm's revenue function. More specifically, consider the following two economic productivity change indexes, Pr^0 and Pr^1, defined as follows:[25]

$$Pr^t \equiv r^1(p^t, x^t)/r^0(p^t, x^t), \qquad t = 0, 1. \tag{47}$$

Thus Pr^t is a measure of the outward shift in the technology going from period 0 to period 1, using the period t output price vector p^t and the period t input quantity vector x^t as reference vectors.

If there has been an efficiency improvement due to a process innovation or improved managerial practices, then Pr^0 and Pr^1 will be greater than 1.

We shall assume competitive profit maximizing behavior on the part of the firm in each period. This implies both revenue maximizing and cost minimizing behavior. Thus we have observed period t revenue, $p^t \cdot y^t$, equal to maximized revenues; i.e., we have

$$p^t \cdot y^t = r^t(p^t, x^t), \qquad t = 0, 1. \tag{48}$$

If r^t is differentiable at p^t, x^t, then by a result due to Hotelling [1932; 594], we have

$$y^t = \nabla_p r^t(p^t, x^t), \qquad t = 0, 1. \tag{49}$$

Furthermore, under the assumption of competitive profit maximizing behavior, for $t = 0, 1$, the period t observed input vector x^t must be a solution to the profit maximization problem:

$$\max_x \{r^t(p^t, x) - w^t \cdot x\}. \tag{50}$$

[24]This shift in the production function approach dates back to Tinbergen [1942] and Solow [1957]. Jorgenson and Griliches [1967; 253] seem to have been the first to note that productivity change could also be defined in terms of shifts in the dual profit function.

[25]This type of theoretical productivity index was defined by Diewert [1983b; 1063].

Assuming that $x^t \gg 0_N$, the first order necessary conditions for (50) are:

$$(51) \qquad w^t = \nabla_x r^t(p^t, x^t), \qquad t = 0, 1.$$

We shall assume that the firm's period t revenue function r^t has the following functional form which generalizes (35):

$$(52) \quad r^t(p,x) \equiv \sigma_t(p^T A p x^T C x + \alpha^t \cdot p \beta^t \cdot x p^T B^t x)^{1/2}, \quad A = A^T,\ C = C^T,\ t = 0, 1,$$

where σ_t is a positive number, A, B^t and C are parameter matrices and α^t and β^t are parameter vectors. Note that the symmetric parameter matrices A and C are constant over the two periods but the other parameters are allowed to vary.

THEOREM 4. *Suppose that the parameters of the revenue functions r^0 and r^1 defined by (52) satisfy restrictions (53) or (54):*

$$(53) \qquad p^{0T} B^0 = 0_N^T; \quad B^0 x^0 = 0_M; \quad \alpha^0 \cdot p^1 = 0 \text{ and } \beta^0 \cdot x^1 = 0 \text{ or}$$

$$(54) \qquad p^{1T} B^1 = 0_N^T; \quad B^1 x^1 = 0_M; \quad \alpha^1 \cdot p^0 = 0 \text{ and } \beta^1 \cdot x^0 = 0.$$

Suppose further that the observed period t input vector x^t is strictly positive for $t = 0, 1$. Then assuming competitive profit maximizing behavior for periods 0 and 1, the theoretical productivity indexes Pr^0 and Pr^1 defined above by (47) are both equal to the Fisher productivity index Pr_F defined by (6); i.e., we have

$$(55) \qquad Pr^0 = Pr_F(p^0, p^1, y^0, y^1, w^0, w^1, x^0, x^1) = Pr^1 = \sigma_1/\sigma_0.$$

Proof: Under assumptions (53) or (54), it can be verified that

$$(56) \qquad r^t(p^t, x^t) = \sigma_t(p^{tT} A p^t x^{tT} C x^t)^{1/2}, \qquad t = 0, 1.$$

Hence using definitions (47), we have

$$(57) \qquad Pr^0 = Pr^1 = \sigma_1/\sigma_0.$$

From (49), we have for $t = 0, 1$:

$$(58) \qquad y^t = \nabla_p r^t(p^t, x^t) = \sigma_t^2 A p^t x^{tT} C x^t / r^t(p^t, x^t),$$

differentiating definitions (52) and using (53) or (54). Similarly, from (51) we have for $t = 0, 1$:

$$(59) \qquad w^t = \nabla_x r^t(p^t, x^t) = \sigma_t^2 C x^t p^{tT} A p^t / r^t(p^t, x^t).$$

Using (59) and (56), we have

$$(60) \qquad w^t \cdot x^t = r^t(p^t, x^t), \qquad t = 0, 1.$$

The square of the Fisher output index is:

$$[Q_F(p^0, p^1, y^0, y^1)]^2 = (p^0 \cdot y^1 / p^0 \cdot y^0)/(p^1 \cdot y^0 / p^1 \cdot y^1)$$

$$= [\sigma_1^2 p^{0T} A p^1 x^{1T} C x^1 / r^1(p^1, x^1) p^0 \cdot y^0]/[\sigma_0^2 p^{1T} A p^0 x^{0T} C x^0 / r^0(p^0, x^0) p^1 \cdot y^1]$$

using (58)

$$(61) \quad = (\sigma_1^2 x^{1T} C x^1)/(\sigma_0^2 x^{0T} C x^0) \qquad \text{using } A = A^T \text{ and (48).}$$

The square of the Fisher input index is:

$$[Q_F^*(w^0, w^1, x^0, x^1)]^2 = (w^0 \cdot x^1 / w^0 \cdot x^0)/(w^1 \cdot x^0 / w^1 \cdot x^1)$$

$$= [\sigma_0^2 x^{1T} C x^0 p^{0T} A p^0 / r^0(p^0, x^0) w^0 \cdot x^0]/[\sigma_1^2 x^{0T} C x^1 p^{1T} A p^1 / r^1(p^1, x^1) w^1 \cdot x^1]$$

using (59)

$$= [\sigma_0^2 p^{0T} A p^0 / r^0(p^0, x^0)^2]/[\sigma_1^2 p^{1T} A p^1 / r^1(p^1, x^1)^2]$$

using $C = C^T$ and (60)

$$= (\sigma_0^2 p^{0T} A p^0 / \sigma_0^2 p^{0T} A p^0 x^{0T} C x^0)/(\sigma_1^2 p^{1T} A p^1 / \sigma_1^2 p^{1T} A p^1 x^{1T} C x^1)$$

using (56)

$$(62) \quad = x^{1T} C x^1 / x^{0T} C x^0.$$

Take the ratio of (61) to (62), take the positive square root and we obtain:

$$(63) \qquad Pr_F(p^0, p^1, y^0, y^1, w^0, w^1, x^0, x^1) = \sigma_1/\sigma_0.$$

The equalities (57) and (63) imply (55). Q.E.D.

The restrictions (53) are consistent with $r^0(p, x)$ being flexible at p^0, x^0 while the restrictions (54) are consistent with $r^1(p, x)$ being flexible at (p^1, x^1). Thus since the Fisher productivity index Pr_F is exact for the theoretical productivity indexes Pr^0 and Pr^1, Pr_F is a ***superlative*** measure of productivity change.[26]

[26] This terminology is analogous to Diewert's [1976a; 117] definition of a superlative index number formula.

Thus Theorem 4 provides a strong economic justification for the use of the Fisher productivity index.[27] However, the revenue functions r^t in the theorem correspond to constant returns to scale technologies. Thus if we want to apply the theorem to a diminishing returns to scale technology, it is necessary to add an artificial fixed input and set it equal to one (i.e., $x^t_{N+1} \equiv 1$). The corresponding period t price, w^t_{N+1}, must be set equal to the firm's period t pure profits.

In the remainder of this paper, we follow the example of Caves, Christensen and Diewert [1982b] and use the distance function approach for defining theoretical input, output and productivity indexes. The reader who is mainly interested in productivity indexes can skip ahead to Section 8.

6. An Economic Approach to Indexes of Real Input

As usual, let $w^t \equiv (w^t_1, \ldots, w^t_N)$ be the input price vector for a firm in period t and let $x^t \equiv (x^t_1, \ldots, x^t_N)$ be the corresponding positive input quantity vector for $t = 0, 1$.[28]

In order to aggregate x^0 and x^1, Caves, Christensen and Diewert [1982b; 1395–1399] developed the concept of the Malmquist [1953] input index, which was first defined geometrically in the two input case by Moorsteen [1961; 460] and was perhaps verbally stated by Hicks [1961; footnote 4] [1981; 256]. Caves, Christensen and Diewert [1982b; 1398] found that an average of two theoretical Malmquist input indexes could be approximated rather well under certain conditions by a translog or Törnqvist index of inputs Q^*_T defined as:

$$(64) \qquad Q^*_T(w^0, w^1, x^0, x^1) \equiv \prod_{i=1}^{N} (x^1_i/x^0_i)^{(1/2)(s^0_i + s^1_i)}$$

where $s^t_i \equiv p^t_i x^t_i / p^t \cdot x^t$ is the period t cost share for input i.

Our purpose in this section is to provide an analogous justification for the Fisher input index Q^*_F defined by:

$$(65) \qquad Q^*_F(w^0, w^1, x^0, x^1) \equiv (p^1 \cdot x^1 p^0 \cdot x^1 / p^0 \cdot x^1 p^0 \cdot x^0)^{1/2}.$$

[27] Diewert [1976a; 126–130] provided an economic justification for the use of Pr_F under rather strong separability assumptions; i.e., the period 0 transformation function had the form $g(y) = f(x)$ while the period 1 transformation function had the form $g(y) = (1+\tau)f(x)$. See Blackorby, Primont and Russell [1978] on separability concepts.

[28] Alternatively, t could index two different firms in the same industry.

As in Section 4, let f denote a firm's production function, let $x \equiv (x_1, \ldots, x_N)$ denote a positive input vector and let $y \equiv (y_1, \ldots, y_M)$ denote a nonnegative output vector. In order to define our theoretical indexes of real input, it is first necessary to define the firm's *input distance* (or deflation) *function* D as follows:[29]

$$(66) \qquad D(y, x) \equiv \max_{\delta > 0}\{\delta : y_1 \le f(y_2, \ldots, y_M, x_1/\delta, x_2/\delta, \ldots, x_N/\delta)\}.$$

Thus $D(y, x)$ is the maximal deflation factor δ^* which will just put the output vector y and the deflated input vector x/δ^* onto the boundary of the feasible production set.

It is easy to verify that $D(y, x)$ defined by (66) will be linearly homogeneous in the components of x; i.e., for a scalar $\lambda > 0$, we have

$$(67) \qquad D(y, \lambda x) = \lambda D(y, x).$$

Hence if D is twice continuously differentiable with respect to the components of x, Euler's theorem on homogeneous functions may be used to establish the following identities:

$$(68) \qquad D(y, x) = x^T \nabla_x D(y, x);$$

$$(69) \qquad \nabla^2_{xx} D(y, x) x = 0_N;$$

$$(70) \qquad \nabla^2_{yx} D(y, x) x = \nabla_y D(y, x).$$

If the production function f is linearly homogeneous, so that there are constant returns to scale, then it can be shown that $D(y, x)$ defined by (66) is homogeneous of degree minus one in the components of y; i.e., for $\lambda > 0$, we have:

$$(71) \qquad D(\lambda y, x) = \lambda^{-1} D(y, x).$$

Hence if D is twice continuously differentiable, Euler's theorem on homogeneous functions may be used to establish the following identities:

$$(72) \qquad D(y, x) = -y^T \nabla_y D(y, x);$$

$$(73) \qquad \nabla^2_{yy} D(y, x) y = -2 \nabla_y D(y, x);$$

$$(74) \qquad y^T \nabla^2_{yx} D(y, x) = -\nabla^T_x D(y, x).$$

[29] Some minimal regularity conditions on f will be required to ensure that the maximum in (66) exists. For the one output case ($M = 1$), appropriate regularity conditions and duality theorems may be found in Blackorby, Primont and Russell [1978; 25–26] and in Diewert [1982; 559–561].

In the remainder of this section, we shall assume that production functions are linearly homogeneous so that constant returns to scale prevail.[30]

We first want to find a functional form for $D(y,x)$ which is *flexible* at y^*, x^*; i.e., given an arbitrary twice continuously differentiable input distance function $D^*(y,x)$ which is homogeneous of degree one in x and homogeneous of degree minus one in y, we want to find a functional form for D such that the following equalities hold:

$$D(y^*,x^*) = D^*(y^*,x^*) = 1; \tag{75}$$

$$\nabla_y D(y^*,x^*) = \nabla_y D^*(y^*,x^*); \tag{76}$$

$$\nabla_x D(y^*,x^*) = \nabla_x D^*(y^*,x^*); \tag{77}$$

$$\nabla^2_{yy} D(y^*,x^*) = \nabla^2_{yy} D^*(y^*,x^*); \tag{78}$$

$$\nabla^2_{xx} D(y^*,x^*) = \nabla^2_{xx} D^*(y^*,x^*); \tag{79}$$

$$\nabla^2_{yx} D(y^*,x^*) = \nabla^2_{yx} D^*(y^*,x^*). \tag{80}$$

We have set $D^*(y^*,x^*) = 1$, which means that (y^*,x^*) is on the production surface. The equalities (76)–(80) mean that D is to have the same first and second order partial derivatives as D^* at the point (y^*,x^*).

For a positive output vector $y \equiv (y_1,\ldots,y_M)$, we shall define y^{-1} to be the vector $(y_1^{-1},\ldots,y_M^{-1})$. Now consider the following functional form for D:

$$D(y,x) \equiv [(y^TAy)^{-1}x^TCx + \alpha\cdot y^{-1}\beta\cdot x(y^{-1})^TBx]^{1/2}, \tag{81}$$

$$A = A^T,\ C = C^T$$

where A, B and C are parameter matrices and α and β are parameter vectors.

It is easy to verify that $D(y,x)$ defined by (81) is homogeneous of degree one in x and homogeneous of degree minus one in y. Thus both D and D^* satisfy the restrictions (68)–(74) at (y^*,x^*).

THEOREM 5. *Let $y^* \gg 0_M$ and $x^* \gg 0_N$. Then for every pair of vectors α and β such that*

$$\alpha\cdot y^{*-1} \neq 0, \qquad \beta\cdot x^* \neq 0, \tag{82}$$

[30]However, the results of this section are still valid for the nonconstant returns to scale case: we need only add an artificial output $M+1$ whose quantity is always equal to one; i.e., define $y_{M+1} \equiv 1$ and redefine the output vector as $y \equiv (y_1,\ldots,y_M,y_{M+1})$. Note that the restriction (72) is implied by (68), (70) and (74).

there exist symmetric matrices A and C such that

$$y^{*T}Ay^* = 1, \qquad x^{*T}Cx^* = 1 \tag{83}$$

and a matrix B such that

$$y^{*-1T}B = 0_N^T, \qquad Bx^* = 0_M \tag{84}$$

and D defined by (81) is flexible at y^*, x^*.

Proof: Define A, B and C in terms of the derivatives of D^* as follows:

$$A \equiv -\nabla^2_{yy}D^*(y^*,x^*) + 3\nabla_y D^*(y^*,x^*)\nabla_y^T D^*(y^*,x^*); \tag{85}$$

$$C \equiv \nabla^2_{xx}D^*(y^*,x^*) + \nabla_x D^*(y^*,x^*)\nabla_x^T D^*(y^*,x^*); \tag{86}$$

$$B \equiv (\alpha\cdot y^{*-1})^{-1}(\beta\cdot x^*)^{-1}\hat{y}^{*2}\{-\nabla^2_{yx}D^*(y^*,x^*) + \nabla_y D^*(y^*,x^*)\nabla_x^T D^*(y^*,x^*)\} \tag{87}$$

where $\hat{y}^*$ is a diagonal matrix with the elements of y^* on the main diagonal and $\hat{y}^{*2} \equiv \hat{y}^*\hat{y}^*$.

We first show that A, B and C defined by (85)–(87) satisfy the restrictions (83) and (84). Postmultiply both sides of (85) by y^* and use (72) and (73) applied to D^* to obtain the following equation:

$$Ay^* = -\nabla_y D^*(y^*,x^*). \tag{88}$$

Now premultiply both sides of (88) by y^{*T}, use (72) applied to D^* and use $D^*(y^*,x^*) = 1$ to obtain the first equality in (83).

Postmultiply both sides of (86) by x^*, use (68) and (69) applied to D^* and obtain:

$$Cx^* = \nabla_x D^*(y^*,x^*). \tag{89}$$

Premultiply both sides of (89) by x^{*T}, use (68) applied to D^* and use $D^*(y^*,x^*) = 1$ to obtain the second equality in (83).

Premultiply both sides of (87) by y^{*-1T}. Note that $y^{*-1T}\hat{y}^{*2} = \hat{y}^{*T}$. Now use (72) and (74) applied to D^* and use $D^*(y^*,x^*) = 1$ to obtain the first set of equations in (84).

Postmultiply both sides of (87) by x^*. Using (68) and (70) applied to D^* and $D^*(y^*,x^*) = 1$, we obtain the second set of equations in (84).

Using the definition of D, (81), and the restrictions (83) and (84), it can be verified that $D(y^*,x^*) = 1$. Hence (75) holds.

Now differentiate the D defined by (81) and evaluate the first order partial derivatives at y^*, x^*. Using the restrictions (83) and (84), we obtain equations (90) and (91) below:

$$\begin{aligned}(90)\qquad \nabla_y D(y^*,x^*) &= -Ay^* \\ &= \nabla_y D^*(y^*,x^*) \quad \text{using (88) and}\end{aligned}$$

$$\begin{aligned}(91)\qquad \nabla_x D(y^*,x^*) &= Cx^* \\ &= \nabla_x D^*(y^*,x^*) \quad \text{using (89).}\end{aligned}$$

Thus (76) and (77) are satisfied.

The matrices of second order derivatives of $D(y^*,x^*)$ are given by equations (92)–(94) below, using the restrictions (83) and (84):

$$(92)\qquad \nabla^2_{yy} D(y^*,x^*) = -A + 3(Ay^*)(Ay^*)^T;$$

$$(93)\qquad \nabla^2_{xx} D(y^*,x^*) = C - (Cx^*)(Cx^*)^T;$$

$$(94)\qquad \nabla^2_{yx} D(y^*,x^*) = -(\alpha\cdot y^{*-1})(\beta\cdot x^*)\hat{y}^{*-2}B - (Ay^*)(Cx^*)^T.$$

If we equate the second order derivatives of D to the corresponding second order derivatives of D^*, then using (76), (77), (90) and (91), it can be seen that the resulting three matrix equations are equivalent to equations (85)–(87), which were used to define A, B and C. Thus equations (78)–(80) are also satisfied and hence D defined by (81) is flexible at y^*, x^*. Q.E.D.

We now allow the firm's production function to depend on time; i.e., in period t, the production function is f^t. We define an ***input deflation function*** D^t for each production function f^t as follows: for $t = 0, 1$,

$$(95)\qquad D^t(y,x) \equiv \max_{\delta>0}\{\delta : y_1 \le f^t(y_2,\ldots,y_M, x_1/\delta, x_2/\delta,\ldots,x_N/\delta)\}.$$

In each period t, we assume that the observed output vector y^t and the observed input vector x^t are on the period t production surface, so that we have:

$$(96)\qquad D^t(y^t,x^t) = 1, \qquad t = 0,1.$$

Following Caves, Christensen and Diewert [1982b; 1396], we define two Malmquist theoretical input indexes Q^{*0} and Q^{*1} as follows:

$$\begin{aligned}(97)\qquad Q^{*0}(x^0,x^1) &\equiv D^0(y^0,x^1)/D^0(y^0,x^0) \\ &= D^0(y^0,x^1) \qquad \text{using (96);}\end{aligned}$$

$$\begin{aligned}(98)\qquad Q^{*1}(x^0,x^1) &\equiv D^1(y^1,x^1)/D^1(y^1,x^0) \\ &= 1/D^1(y^1,x^0) \qquad \text{using (96).}\end{aligned}$$

Let us interpret Q^{*0}. Let $\delta^0 = D^0(y^0,x^1)$. Then using definition (95), we have $y_1^0 = f^0(y_2^0,\ldots,y_M^0,x_1^1/\delta^0,\ldots,x_N^1/\delta^0)$ and by (96), we have $y_1^0 = f^0(y_2^0,\ldots,y_M^0,x_1^0,\ldots,x_N^0)$. Thus the deflated input vector x^1/δ^0 is equivalent to the period 0 input vector x^0 from the viewpoint of the period 0 technology and the deflation factor $\delta^0 = D^0(y^0,x^1) = Q^{*0}(x^0,x^1)$ is a natural measure of the size of x^1 relative to x^0; if $\delta^0 > 1$, then x^1 is bigger than x^0; if $\delta^0 = 1$, then x^0 is equivalent to x^1; and if $\delta^0 < 1$, then x^1 is less than x^0.

The interpretation for (98) is similar but the comparisons are made using the period 1 technology. Let $\delta^1 = 1/D^1(y^1,x^0)$, or $D^1(y^1,x^0) = 1/\delta^1$. Then by definition (95), we have $y_1^1 = f^1(y_2^1,\ldots,y_M^1,\delta^1x_1^0,\ldots,\delta^1x_N^0)$ and by (96), we have $y_1^1 = f^1(y_2^1,\ldots,y_M^1,x_1^1,\ldots,x_N^1)$. Thus δ^1x^0 is equivalent to the period 1 input vector x^1 from the viewpoint of the period 1 technology and $\delta^1 = 1/D^1(y^1,x^0) = Q^{*1}(x^0,x^1)$ is another natural measure for the size of x^1 relative to x^0.

The firm's period t cost function C^t may be defined in terms of the period t production function f^t as follows: for a target output vector $y \equiv (y_1,\ldots,y_M)$ and a given vector of positive input prices w, define

$$(99)\qquad C^t(y,w) \equiv \min_x\{w\cdot x : y_1 \le f^t(y_2,\ldots,y_M,x)\}, \qquad t = 0,1.$$

For our next theorem, we will require the hypothesis of cost minimizing behavior on the part of the firm for periods 0 and 1; i.e., if w^t is the observed input price vector for period t and x^t, y^t are the observed input and output vectors respectively for period t, then we assume:

$$(100)\qquad w^t\cdot x^t = C^t(y^t,w^t), \qquad t = 0,1.$$

Assuming that $D^t(y^t,x^t)$ is differentiable with respect to the components of w, Caves, Christensen and Diewert [1982b; 1397] show that the following equalities hold:

$$(101)\qquad w^t/w^t\cdot x^t = \nabla_x D^t(y^t,x^t), \qquad t = 0,1.$$

We now assume that the period t input deflation function D^t has the following functional form:

$$(102)\qquad D^t(y,x) \equiv [(y^TA^ty)^{-1}x^TCx + \alpha^t\cdot y^{-1}\beta^t\cdot xy^{-1T}B^tx]^{1/2}, \qquad t = 0,1,$$

where the matrices A^t and C are symmetric.

Note that the D^t defined by (102) are generalizations of the flexible D defined by (81): in (102), the matrices A^t and B^t and the vectors α^t and β^t are now allowed to depend on time, whereas in (81), these parameter matrices and vectors were fixed. However, note that the parameter matrix C in (102) is fixed across the two time periods.

THEOREM 6. *Suppose the parameters of D^0 and D^1 defined by (102) satisfy the following restrictions:*

$$(103)\qquad (y^{tT}A^t y^t)^{-1} x^{tT} C x^t = 1, \qquad t = 0,1;$$

$$(104)\qquad (y^0)^{-1T} B^0 = 0_N^T;$$

$$(105)\qquad \alpha^1 \cdot (y^1)^{-1} = 0;$$

where the column vectors $(y^t)^{-1} \equiv [(y_1^t)^{-1}, (y_2^t)^{-1}, \ldots, (y_M^t)^{-1}]^T$ for $t = 0,1$. Suppose also that the firm is engaging in competitive cost minimizing behavior in the two periods so that the relations (96), (100) and (101) hold. Then the Fisher ideal input index Q_F^* is equal to each of the theoretical input indexes Q^{*0} and Q^{*1} defined by (97) and (98); i.e., we have

$$(106)\qquad Q_F^*(w^0, w^1, x^0, x^1) = Q^{*0}(x^0, x^1) = Q^{*1}(x^0, x^1).$$

Proof: Note that the restrictions (103)–(105) imply that $D^0(y^0, x^0) = D^1(y^1, x^1) = 1$. Now write the square of the Fisher input index as follows:

$$\begin{aligned}
Q_F^*(w^0, w^1, x^0, x^1)^2 &= [(w^0/w^0 \cdot x^0) \cdot x^1]/[(w^1/w^1 \cdot x^1) \cdot x^0] \\
&= [x^{1T} \nabla_x D^0(y^0, x^0)]/[x^{0T} \nabla_x D^1(y^1, x^1)] \\
&\qquad \text{using (101)} \\
&= (x^{1T} C x^0 / y^{0T} A^0 y^0)/(x^{0t} C x^1 / y^{1T} A^1 y^1) \\
&\qquad \text{using (102)–(105)} \\
(107)\qquad &= (y^{0T} A^0 y^0)^{-1}/(y^{1T} A^1 y^1)^{-1} \\
&\qquad \text{using } C = C^T \\
&= (y^{0T} A y^0)^{-1} x^{1T} C x^1 / (y^{1T} A^1 y^1)^{-1} x^{1T} C x^1 \\
&= (y^0 A y^0)^{-1} x^{1T} C x^1 \\
&\qquad \text{using (103)} \\
&= [D^0(y^0, x^1)]^2 \\
&\qquad \text{using (102) and (104)} \\
(108)\qquad &= [Q^{*0}(x^0, x^1)]^2 \\
&\qquad \text{using (97)} \\
&= (y^{0T} A^0 y^0)^{-1} x^{0T} C x^0 / (y^{1T} A^1 y^1)^{-1} x^{0T} C x^0 \\
&\qquad \text{using (107)} \\
&= 1/(y^{1T} A^1 y^1)^{-1} x^{0T} C x^0 \\
&\qquad \text{using (103)} \\
&= 1/[D^1(y^1, x^0)]^2 \\
&\qquad \text{using (102) and (105)} \\
(109)\qquad &= [Q^{*1}(x^0, x^1)]^2 \\
&\qquad \text{using (98).}
\end{aligned}$$

Taking square roots of (108) and (109) yields (106). Q.E.D.

COROLLARY. *The above theorem holds if the restrictions (104) and (105) are replaced by the following restrictions:*

$$(110)\qquad (y^1)^{-1T} B^1 = 0_N^T;$$

$$(111)\qquad \alpha^0 \cdot (y^0)^{-1} = 0.$$

The proof follows by a series of computations similar to (107) through (109).

We note that the restrictions (104) are consistent with D^0 being flexible at (y^0, x^0) while the restrictions (110) are consistent with D^1 being flexible at (y^1, x^1).

Note that the above theorem does not require optimizing behavior with respect to outputs.

Theorem 6 is a Fisher input index counterpart to the translog input index justification derived by Caves, Christensen and Diewert [1982b; 1398] who showed that under certain conditions, a geometric mean of Q^{*0} and Q^{*1} was equal to the translog input index Q_T^* defined by (64). Our present result is perhaps marginally better in that we no longer have to take a geometric mean of Q^{*0} and Q^{*1}.

We turn now to a parallel discussion of output indexes.

7. An Economic Approach to Indexes of Real Output

Our theoretical treatment of output indexes follows that of Caves, Christensen and Diewert [1982b; 1399–1401]. We now represent the technology in period t by means of an input requirements function g^t where $x_1 = g^t(y_1, \ldots, y_M, x_2, \ldots, x_N)$ is the minimum amount of input one required to produce the vector of outputs $y \equiv (y_1, \ldots, y_M)$ given that x_n units of input n are available for $n = 2, 3, \ldots, N$.

The period t *output deflation or distance function* d^t is defined as follows:

$$(112)\qquad d^t(y, x) \equiv \min_{\delta > 0}\{\delta : g^t(y_1/\delta, \ldots, y_M/\delta, x_2, \ldots, x_N) \le x_1\}, \qquad t = 0,1.$$

It is straightforward to show that $d^t(y, x)$ must be homogeneous of degree one in the components of y. Moreover, if the period t technology exhibits constant returns to scale, then g^t is homogeneous of degree one in its arguments and it can be shown that $d^t(y, x)$ must be homogeneous of degree minus one in the components of x. Thus in the constant returns to scale case, the regularity conditions on the output deflation functions d^t are the reverse of the regularity conditions on the input deflation function $D(y,x)$ which was defined in the previous section. Thus in the differentiable case, conditions analogous to (67)–(74) hold for d^t except that the roles of x and y are interchanged.

For $x \gg 0$, define the vector $x^{-1} \equiv (x_1^{-1},\ldots,x_N^{-1})$ and define the following distance function:

(113) $d(y,x) \equiv [y^T Ay(x^T Cx)^{-1} + \alpha\cdot y\beta\cdot x^{-1} y^T Bx^{-1}]^{1/2}, \quad A = A^T,\ C = C^T$

where A, B and C are matrices of parameters and α and β are vectors of parameters.

The following theorem is analogous to Theorem 5 and can be proven in the same manner.

THEOREM 7. *Let $y^* \gg 0_M$ and $x^* \gg 0_N$. Then for every α and β such that*

(114) $$\alpha\cdot y^* \neq 0, \qquad \beta\cdot x^{*-1} \neq 0,$$

there exist symmetric matrices A and C such that

(115) $$y^{*T}Ay^* = 1, \qquad x^{*T}Cx^* = 1$$

and a matrix B such that

(116) $$y^{*T}B = 0_N^T, \qquad Bx^{*-1} = 0_M$$

and the output deflation function d defined by (113) is flexible at y^*, x^*.

If we wish to relax the assumption of a constant returns to scale technology, then we need only add an extra input to the list of inputs and fix its level; i.e., define $x_{N+1} \equiv 1$. Then $d(y,x)$ defined by (113) (where x is now an $N+1$ dimensional vector) will be a flexible output deflation function in the class of nonconstant returns to scale technologies.

We shall assume that the observed period t output and input vectors, y^t and x^t respectively, are efficient relative to the firm's period t technology; i.e., we assume that $x_1^t = g^t(y^t, x_2^t,\ldots,x_N^t)$ for $t=0,1$. Then by (112), we shall have:

(117) $$d^t(y^t,x^t) = 1, \qquad t=0,1.$$

Following the example of Caves, Christensen and Diewert [1982b; 1400], we use the output deflation functions d^t in order to define the following theoretical Malmquist[31] output indexes Q^0 and Q^1:

(118) $$\begin{aligned} Q^0(y^0,y^1) &\equiv d^0(y^1,x^0)/d^0(y^0,x^0) \\ &= d^0(y^1,x^0) \qquad \text{using (117);} \end{aligned}$$

(119) $$\begin{aligned} Q^1(y^0,y^1) &\equiv d^1(y^1,x^1)/d^1(y^0,x^1) \\ &= 1/d^1(y^0,x^1) \qquad \text{using (117).} \end{aligned}$$

[31]The basic idea of these indexes appears to be in Hicks [1961] [1981; 256] and they are defined geometrically in the two output case by Moorsteen [1961; 452].

An interpretation for the theoretical index defined by (118) runs as follows. Let $\delta^0 = d^0(y^1,x^0)$. Then by definition (112), we have $x_1^0 = g^0(y^1/\delta^0, x_2^0,\ldots,x_N^0)$. From (117) we have $x_1^0 = g^0(y^0,x_2^0,\ldots,x_N^0)$. Thus the deflated period 1 output vector y^1/δ^0 is "equivalent" using the period 0 technology to the period 0 output vector y^0, and the deflation factor $\delta^0 = d^0(y^1,x^0) = Q^0(y^0,y^1)$ is a natural scalar measure of the size of y^1 relative to y^0.

Similarly, let $1/d^1 = d^1(y^0,x^1)$. Then by definition (112), $x_1^1 = g^1(y^0\delta^1, x_2^1,\ldots,x_N^1)$ and by (117), $x_1^1 = g^1(y^1,x_2^1,\ldots,x_N^1)$. Thus $\delta^1 y^0$ is "equivalent" to y^1 and $\delta^1 = 1/d^1(y^0,x^1) = Q^1(y^0,y^1)$ is a natural scalar measure of the size of y^1 relative to y^0.

The firm's *period t revenue function* $r^t(p,x)$ may be defined in terms of the factor requirements function g^t as follows: given a vector of inputs $x \equiv (x_1,\ldots,x_N)$ and a vector of positive output prices $p \equiv (p_1,\ldots,p_M)$, define

(120) $$r^t(p,x) \equiv \max_y\{p\cdot y : g^t(y,x_2,\ldots,x_N) \le x_1\}, \qquad t=0,1.$$

For our next theorem, we shall require the hypothesis of revenue maximizing behavior on the part of the firm for periods 0 and 1; i.e., if p^t is the observed output price vector for period t and x^t, y^t are the observed input and output vectors respectively for period t, then we shall assume:

(121) $$p^t\cdot y^t = r^t(p^t,x^t), \qquad t=0,1.$$

Assuming that $d^t(y^t,x^t)$ was differentiable with respect to the components of y, Caves, Christensen and Diewert [1982b; 1401] showed that the following equalities hold:

(122) $$p^t/p^t\cdot y^t = \nabla_y d^t(y^t,x^t), \qquad t=0,1.$$

We may now prove the following theorem which is an exact counterpart to Theorem 6 in the previous section.

THEOREM 8. *Suppose that the firm's output deflation function in period t, d^t, has the following functional form for $t=0,1$:*

(123) $$d^t(y,x) \equiv [y^T Ay(x^T C^t x)^{-1} + \alpha^t\cdot y\beta^t\cdot x^{-1} y^T B^t x^{-1}]^{1/2}$$

where A and C^t are symmetric matrices and the parameters in (123) satisfy the following restrictions:

(124) $$y^{tT}Ay^t(x^{tT}C^t x^t)^{-1} = 1, \qquad t=0,1$$

and either the following restrictions are satisfied:

(125) $$B^0(x^0)^{-1} = 0_M; \qquad \beta^1\cdot(x^1)^{-1} = 0,$$

or the following restrictions are satisfied:

$$B^1(x^1)^{-1} = 0_M; \qquad \beta^0 \cdot (x^0)^{-1} = 0. \tag{126}$$

Suppose also that the firm is engaging in competitive revenue maximizing behavior in periods 0 and 1 so that the relations (121), (122) and (117) hold. Then the Fisher output index Q_F defined by (4) is equal to each of the theoretical indexes Q^0 and Q^1 defined by (118) and (119); i.e., we have:

$$Q_F(p^0, p^1, y^0, y^1) = Q^0(y^0, y^1) = Q^1(y^0, y^1). \tag{127}$$

The proof is analogous to the proof of Theorem 6 in the previous section. We note that the restrictions (125) are consistent with d^0 being flexible at (y^0, x^0) while the restrictions (126) are consistent with d^1 being flexible at (y^1, x^1).

Theorem 8 is a Fisher index analogue to Theorem 2 in Caves, Christensen and Diewert [1982b; 1401] which showed that the translog or Törnqvist output index Q_T defined by (5) was equal to the geometric mean of Q^0 and Q^1 provided that the firm's distance functions d^t had the translog functional form with identical quadratic coefficients for the second order terms in $\ln y_1, \ldots, \ln y_M$. Note that our present result does not require us to take a geometric mean of Q^0 and Q^1. Note also that the matrix A which has the quadratic terms in y in $d^0(y,x)$ and $d^1(y,x)$ defined by (123) is constant across time periods. Thus in both Theorem 8 and Theorem 2 of Caves, Christensen and Diewert, the technologies in the two time periods cannot be completely different.

8. An Alternative Economic Approach to Productivity Indexes

We can draw on the material of the previous two sections to define productivity or efficiency indexes.

One approach would be to define a productivity index to be the ratio of the Malmquist output index $Q^0(y^0, y^1)$ or $Q^1(y^0, y^1)$ defined by (118) and (119) divided by a Malmquist input index $Q^{*0}(x^0, x^1)$ or $Q^{*1}(x^0, x^1)$ defined by (97) or (98). Thus there are four possible theoretical productivity indexes that could be defined in this manner. In the two input, two output case, these theoretical efficiency change indexes were suggested by Moorsteen [1961; 462] and perhaps by Hicks [1961; footnote 4] in the general multiple input and output case.

Rather than follow this Hicks-Moorsteen approach to productivity indexes, we shall follow the approach taken by Caves, Christensen and Diewert [1982b; 1401–1408] and use only the output deflation functions $d^t(y,x)$ defined in the previous section by (112) in order to define theoretical productivity indexes.

We assume that in each period, either the firm's technology is subject to constant returns to scale or it is subject to diminishing returns to scale but the firm's pure profits are imputed to an artificial fixed factor.

Let x^t and y^t be the observed input and output vectors for period t and let $d^t(y,x)$ be the firm's period t output deflation function defined by (112) for $t = 0, 1$. Following Caves, Christensen and Diewert [1982b; 1402], define the following *theoretical productivity indexes* Π^0 and Π^1:

$$\begin{aligned} \Pi^0(x^0, x^1, y^0, y^1) &\equiv d^0(y^1, x^1)/d^0(y^0, x^0) \\ &= d^0(y^1, x^1) \qquad \text{using (117);} \end{aligned} \tag{128}$$

$$\begin{aligned} \Pi^1(x^0, x^1, y^0, y^1) &\equiv d^1(y^1, x^1)/d^1(y^0, x^0) \\ &= 1/d^1(y^0, x^0) \qquad \text{using (117).} \end{aligned} \tag{129}$$

We can interpret (128) as follows. Let $\delta^0 = d^0(y^1, x^1)$. Then by definition (112), $x_1^1 = g^0(y^1/\delta^0, x_2^1, \ldots, x_N^1)$. Thus the deflated period 1 output vector g^1/δ^0 and the period 1 input vector x^1 are on the production surface for period 0. Of course, we also have $x_1^0 = g^0(y^0, x_2^0, \ldots, x_N^0)$ so that the period 0 output vector y^0 and the period 0 input vector x^0 are on the period 0 production surface. If there were a productivity improvement going from period 0 to 1, we would expect that the deflation factor δ^0 would be greater than 1 and $\delta^0 = d^0(y^1, x^1) = \Pi^0(x^0, x^1, y^0, y^1)$ can serve as a measure of the magnitude of the productivity improvement. If $\delta^0 = 1$, then the period 1 output-input combination (y^1, x^1) is on the period 0 production surface and there has been no efficiency improvement.

To interpret (129), let $1/\delta^1 = d^1(y^0, x^0)$. Then by definition (112), $x_1^0 = g^1(y^0\delta^1, x_2^0, \ldots, x_N^0)$. This means that the inflated period 0 output vector $\delta^1 y^0 \equiv (\delta^1 y_1^0, \ldots, \delta^1 y_M^0)$ and the period 0 input vector x^0 are on the production surface for period 1. Thus using the period 0 input vector, the period 1 technology can produce δ^1 times the period 0 output vector. Hence the blow up factor $\delta^1 = 1/d^1(y^0, x^0) = \Pi^1(x^0, x^1, y^0, y^1)$ can serve as an index of the productivity improvement.

In Theorem 9 below, we shall again assume revenue maximizing behavior as in the previous section. Thus (121) and (122) must hold if $d^t(y^t, x^t)$ is differentiable. We shall also assume cost minimizing behavior as in Section 6. Under the assumption of cost minimizing behavior and a constant returns to scale technology, Caves, Christensen and Diewert [1982b; 1403–1404] show that the following relations must hold if d^t is differentiable:

$$\nabla_x d^t(y^t, x^t) = -w^t/w^t \cdot x^t, \qquad t = 0, 1. \tag{130}$$

In the following theorem, we shall use more general versions of the output deflation function $d(y,x)$ defined by (81), which was shown to be a flexible functional form in Theorem 7.

THEOREM 9. *Suppose that the firm's output deflation function in period* t, d^t, *is defined as follows for* $t = 0, 1$:

$$(131)\quad d^t(y,x) \equiv \sigma^t[y^T Ay(x^T Cx)^{-1} + \alpha^t \cdot y\beta^t \cdot x^{-1}y^T B^t x^{-1}]^{1/2}, \quad A = A^T,\ C = C^T$$

where the parameter matrices A, B^t *and* C, *the parameter vectors* α^t *and* β^t *and the parameter scalars* σ^t *satisfy the following restrictions:*

$$(132)\quad \sigma^t[y^{tT}Ay^t(x^{tT}Cx^t)^{-1}]^{1/2} = 1, \qquad t = 0,1$$

and either the following restrictions are satisfied:

$$(133)\quad \begin{array}{l} B^0(x^0)^{-1} = 0_M;\quad y^{0T}B^0 = 0_N^T;\quad \alpha^1 \cdot y^1 = 0; \\ \beta^1 \cdot (x^1)^{-1} = 0;\quad \alpha^0 \cdot y^1\beta^0 \cdot (x^1)^{-1} = 0 \quad \text{and} \quad y^{0T}B^1(x^0)^{-1} = 0 \end{array}$$

or the following restrictions are satisfied:

$$(134)\quad \begin{array}{l} B^1(x^1)^{-1} = 0_M;\quad y^{1T}B^1 = 0_N^T;\quad \alpha^0 \cdot y^0 = 0; \\ \beta^0 \cdot (x^0)^{-1} = 0;\quad \alpha^1 \cdot y^0\beta^1 \cdot (x^0)^{-1} = 0 \quad \text{and} \quad y^{1T}B^0(x^1)^{-1} = 0. \end{array}$$

Suppose also that the firm competitively maximizes revenues given inputs and competitively minimizes costs given outputs in each period.[32] *Then the Fisher productivity index* Pr_F *defined by (6) is equal to each of the theoretical productivity indexes* Π^0 *and* Π^1 *defined by (128) and (129); i.e., we have*

$$(135)\quad Q_F(p^0,p^1,y^0,y^1)/Q_F^*(w^0,w^1,x^0,x^1) = \Pi^0(x^0,x^1,y^0,y^1) = \Pi^1(x^0,x^1,y^0,y^1).$$

Proof: The restrictions (132) and (133) or (132) and (134) imply that d^t defined by (131) satisfies the following restriction:

$$(136)\quad d^t(y^t,x^t) = 1, \qquad t = 0,1.$$

The square of the Fisher productivity index is

$$[Q_F(p^0,p^1,y^0,y^1)/Q_F^*(w^0,w^1,x^0,x^1)]^2$$

$$= \left[[(p^0/p^0 \cdot y^0) \cdot y^1]/[(p^1/p^1 \cdot y^1) \cdot y^0]\right] / \left[[(w^0/w^0 \cdot x^0) \cdot x^1]/[(w^1/w^1 \cdot x^1) \cdot x^0]\right]$$

[32]These assumptions are implied by the assumption of competitive profit maximizing behavior in each period.

$$= [y^1 \cdot \nabla_y d^0(y^0,x^0)/y^0 \cdot \nabla_y d^1(y^1,x^1)]$$
$$/[(-1)x^1 \cdot \nabla_x d^0(y^0,x^0)/(-1)x^0 \cdot \nabla_x d^1(y^1,x^1)]$$

using (122) and (130)

$$= [[(\sigma^0)^2 y^{1T}Ay^0/x^{0T}Cx^0]/[(\sigma^1)^2 y^{0T}Ay^0/x^{1T}Cx^1]]$$
$$/\ [[(\sigma^0)^2 x^{1T}Cx^0 y^{0T}Ay^0(x^{0T}Cx^0)^{-2}]/[(\sigma^1)^2 x^{0T}Cx^1 y^{1T}Ay^1(x^{1T}Cx^1)^{-2}]]$$

differentiating (131) and using (132)-(134)

$$(137)\quad = (\sigma^0)^2/(\sigma^1)^2 \qquad \text{using } A = A^T,\ C = C^T \text{ and (132)}$$
$$= (\sigma^0)^2 y^{1T}Ay^1(x^{1T}Cx^1)^{-1}/(\sigma^1)^2 y^{1T}Ay^1(x^{1T}Cx^1)^{-1}$$
$$= (\sigma^0)^2 y^{1T}Ay^1(x^{1T}Cx^1)^{-1} \qquad \text{using (132)}$$
$$= (\sigma^0)^2[y^{1T}Ay^1(x^{1T}Cx^1)^{-1} + \alpha^0 \cdot y^1\beta^0 \cdot (x^1)^{-1}y^{1T}B^0(x^1)^{-1}]$$

using either (133) or (134)

$$= [d^0(y^1,x^1)]^2 \qquad \text{by definition (131)}$$

$$(138)\quad = (\Pi^0(x^0,x^1,y^0,y^1))^2 \qquad \text{using (128)}$$
$$= (\sigma^0)^2 y^{0T}Ay^0(x^{0T}Cx^0)^{-1}/(\sigma^1)^2 y^{0T}Ay^0(x^{0T}Cx^0)^{-1} \quad \text{using (137)}$$
$$= 1/(\sigma^1)^2 y^{0T}Ay^0(x^{0T}Cx^0)^{-1} \quad \text{using (132)}$$
$$= 1/(\sigma^1)^2[y^{0T}Ay^0(x^{0T}Cx^0)^{-1} + \alpha^1 \cdot y^0\beta^1 \cdot (x^0)^{-1}y^{0T}B^1(x^0)^{-1}]$$

using (133) or (134)

$$= 1/[d^1(y^0,x^0)]^2 \qquad \text{by definition (131)}$$

$$(139)\quad = [\Pi^1(x^0,x^1,y^0,y^1)]^2 \qquad \text{using (129).}$$

Q.E.D.

Taking square roots of (138) and (139) yields (135).

The restrictions (133) are consistent with d^0 being flexible at (y^0,x^0) and the restrictions (134) are consistent with d^1 being flexible at (y^1,x^1). Thus the above Theorem is a fairly close analogue to Theorem 3 of Caves, Christensen and Diewert [1982b; 1404] which showed that the translog productivity index

Pr_F defined by (7) was exactly equal to the geometric mean of the theoretical productivity indexes Π^0 and Π^1 defined by (128) and (129) provided that the distance functions d^0 and d^1 had translog functional forms (with identical second order coefficients). Our present result is perhaps a bit stronger in that we find that Pr_F is equal to Π^0 and Π^1 (i.e., we do not have to take a geometric mean of Π^0 and Π^1).

Theorem 9 provides yet another strong justification for the use of the Fisher productivity index.

9. Conclusion

We have presented a number of justifications for the use of the Fisher output index Q_F, the Fisher input index Q_F^* and the Fisher productivity index $Pr_F \equiv Q_F/Q_F^*$. Theorem 1 presents a strong justification for the use of these indexes from the viewpoint of the test approach.

Using the economic approach to index number theory, Theorem 6 justifies the use of the Fisher input index while Theorem 8 justifies the use of the Fisher output index. Theorems 4 and 9 provide strong justifications for the use of the Fisher productivity index from the viewpoint of the economic approach.[33]

If we compare the test approach to productivity indexes with the economic approach, the following points emerge: (i) the test approach suffers from a lack of consensus on what the appropriate tests or axioms should be; (ii) the economic approach requires the assumption of competitive price taking behavior and the assumption of constant (or diminishing) returns to scale. Thus both approaches have their weaknesses. However, we have presented strong justifications for the use of the Fisher productivity index from *both* viewpoints, which should reduce objections to its use in many contexts.

We conclude by noting that both the economic and test approaches to the measurement of productivity change make assumptions which are often not satisfied in empirical contexts such as: (i) all prices and quantities in the two periods are known with certainty;[34] and (ii) there are no new inputs that are used in period 1 but not in period 0 and there are no new outputs that are produced in period 1. With respect to the new good problem, the economic approach can at least provide a theoretical solution to the problem[35] but the empirical difficulties remain formidable.

[33] Many years ago, Denny [1980; 537] asked whether some of my translog exact aggregation results could be generalized to other functional forms. At the time, I could not answer his question, but now Theorems 4, 6, 8 and 9 show that his question has an affirmative answer.

[34] Students of accounting will know that this assumption is very suspect. For example, consider the problems in estimating user costs for durable capital inputs when there is rapid inflation and tax complications.

[35] In period 0, the quantity of the new good is obviously zero and its price is set equal to that shadow price which would just cause the firm to demand or supply a zero amount of that good in period 0. The basic idea is due to Hicks [1940; 114]; see also Fisher and Shell [1972b; 101] and Diewert [1980; 498–503].